A New Collocation Method for Solving Fractional Integro-di Erential Equations with the Weakly Singular Kernel Based on the Fractional C^{α} Space

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Abstract: In this paper, a new collocation method based on fractional polynomials is proposed for solving fractional integro-di erential equations with weakly singular kernel. For solving the equation, the di culties lie in choosing the space of the exact solution. In this paper, we solve this problem perfectly and propose the concept of the fractional C^{α} space. Meanwhile, a dense subset of this fractional C^{α} space is obtained. Based on the fractional C^{α} space, a strict theory for obtaining the ε-approximate solution is established. Using our method, a small amount of calculation can gain an accuracy satisfying the application requirements. The e ciency of the proposed method is verified by the final numerical experiments through comparing with those reported in *A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-di erential equations with weakly singular kernels*.

Keywords: Collocation method; Fractional integro-di erential equations; Weakly singular kernel; Fractional C^{α} space

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1. Introduction

Fractional integro-di erential equations with a weakly singular kernel are used in modeling di erent physical problems such as heat conduction problem [1], elasticity and fracture mechanics [2], etc.

In this paper, we consider the following fractional integro-di erential equation with a weakly singular kernel

$$\begin{cases} D_C^{\alpha}u(x) = g(x) + p(x)u(x) + \int_0^x (x-s)^{-\beta} u(s)\mathrm{d}s, \alpha > 0, 0 \le \beta < 1, x \in [0,1], \\ u^{(i)}(0) = u_0^{(i)}, i = 0,1,\cdots,n-1, \end{cases} \tag{1}$$

where $u(x)$ is the unknown function, $g(x)$ and $p(x)$ are known continuous functions on $[0,1]$, $u_0^{(i)} (i = 0,1,2,\cdots,n-1)$ are any real numbers, $n = \lceil \alpha \rceil$ is the smallest integer which is not smaller than the real number α and D_C^{α} is the Caputo fractional di erential operator of order α.

The existence and uniqueness results for the solution of fractional integro-di erential equations have been obtained in [3]. Equation (1) has been solved by few methods such as collocation methods [4] and spectral methods [5].

In this work, we first select exactly the space $C^{\alpha}[0,1]$ where the exact solution of Equation (1) lies in, and then we prove that fractional polynomials are a dense subset of $C^{\alpha}[0,1]$. Based on this dense subset, we give a method for obtaining the ε-approximate solution of Equation (1) by solving a system of equations or searching the minimum value. The final numerical examples demonstrate the e ciency of the proposed method through comparing the numerical results with those from [5].

2. Selection of Space

In this section, we will give some basic knowledge about the fractional integral and the Caputo fractional di erential operator. Then we will select exactly the space where the exact solution of Equation (1) lies in.

Definition 2.1.

The Riemann-Liouville fractional integral operator J_0^{α} of order α is given by

$$J_0^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} u(s)\mathrm{d}s, \alpha > 0,$$

Where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}\mathrm{d}x.$

Definition 2.2.

The Caputo fractional differential operator D_C^{α} of order α is given by

$$D_C^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1} u^{(n)}(s)\mathrm{d}s, n-1 < \alpha < n.$$

Denote

$$AC^n[0,1] = \left\{ u(x) \middle| u^{(n-1)}(x) \textit{ is absolute continuous on } [0,1] \right\}.$$

Some properties of the Riemann-Liouville fractional integral operator J_0^α and the Caputo fractional differential operator D_C^α are as follows:

Proposition 2.1.

Let $\alpha>0,\beta>0,\nu>-1$ and let $\lceil\alpha\rceil=n$.

(1) $J_0^\alpha x^\nu=\frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\alpha)}x^{\nu+\alpha}$;

(2) If $u(x)\in L^1[0,1]$, then $J_0^\alpha J_0^\beta u(x)=J_0^{\alpha+\beta}u(x)$;

(3) If $u(x)\in C[0,1]$, then $D_C^\alpha J_0^\alpha u(x)=u(x)$;

(4) If $u(x)\in AC^n[0,1]$, then

$$J_0^\alpha\left(D_C^\alpha u(x)\right)=u(x)-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}x^k.$$

Lemma 2.1.

Assuming that $u\in C[0,1],\alpha_1>0$,

then $J_0^{\alpha_1}u(x)\in C[0,1]$ and $\left\|J_0^{\alpha_1}u\right\|_C\le\frac{1}{\Gamma(1+\alpha_1)}\|u\|_C$.

Proof. According to Theorem 2.6 in [6], it's easy to prove that $J_0^{\alpha_1}\in C[0,1]$. Besides,

$$\left|J_0^{\alpha_1}u(x)\right|=\frac{1}{\Gamma(\alpha_1)}\left|\int_0^x(x-t)^{\alpha_1-1}u(t)\mathrm{d}t\right|$$
$$\le\|u\|_C\frac{1}{\Gamma(\alpha_1)}\int_0^x(x-t)^{\alpha_1-1}\,\mathrm{d}t$$
$$=\|u\|_C\frac{x^{\alpha_1}}{\Gamma(1+\alpha_1)}\le\frac{1}{\Gamma(1+\alpha_1)}\|u\|_C.$$

So the conclusion follows.

To choose the space where the exact solution of Equation (1) lies in, we first change the boundary conditions $u^{(i)}(0)=u_0^{(i)}(i=0,1,\cdots,n-1)$, to 0. Introducing a new unknown function

$$v(x)=u(x)-\phi(x), \tag{2}$$

where $\phi(x)=\sum_{k=0}^{n-1}\frac{u_0^{(k)}}{k!}x^k$. Then Equation (1) is reduced equivalently to the following problem.

$$\begin{cases}\mathbb{L}v(x)=D_C^\alpha v(x)-p(x)v(x)-\int_0^x(x-s)^{-\beta}v(s)\mathrm{d}s=f(x),0\le x\le1,\\ v^{(i)}(0)=0,i=0,1,\cdots,n-1.\end{cases} \tag{3}$$

where $f(x)=g(x)+p(x)\sum_{k=0}^{n-1}\frac{u_0^{(k)}}{k!}x^k$

$+\Gamma(1-\beta)\sum_{k=0}^{n-1}\frac{u_0^{(k)}}{\Gamma(k+2-\beta)}x^{k-\beta+1}$.

Denote $J_0^\alpha C=\left\{J_0^\alpha u(x)\middle|u\in C[0,1],\right.$

$\left.,u^{(i)}(0)=0,i=0,1,\cdots,n-1\right\}$.

Lemma 2.2.

Let v be the exact solution of Equation (3), then $v(x)\in J_0^\alpha C$.

Proof. Noting that $\left|D_C^\alpha v(t)\right|=\left|J_0^{n-\alpha}v^{(n)}(t)\right|<\infty$ and the fact that the integrability of a function is equivalent to the absolute integrability, one has

$$\infty>\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-t)^{n-\alpha-1}\left|v^{(n)}(t)\right|dt>\frac{x^{n-\alpha-1}}{\Gamma(n-\alpha)}\int_0^x\left|v^{(n)}(t)\right|dt.$$

Taking $x=1$, we obtain that $v^{(n)}(t)\in\mathbb{L}^1[0,1]$. So $v(t)$ and $v^{(n-1)}(t)$ are absolutely continuous on $[0,1]$. Hence, Lemma 2.1 shows that $D_C^\alpha v(x)=f(x)+p(x)v(x)+\Gamma(1-\beta)J_0^{1-\beta}v(x)\in C[0,1]$. Noting that $v^{(n-1)}(x)\in AC^n[0,1]$ and using the fourth assertion of Property 2.1, we have $v(x)=J_0^\alpha D_C^\alpha v(x)\in J_0^\alpha C$.

We need the following properties of $J_0^\alpha C$.

Proposition 2.2.

Let $\omega\in J_0^\alpha C$, then

(1) $\omega^{(i)}(0)=0(i=0,1,2,\cdots,n-1),D_C^\alpha\omega\in C[0,1]$;

(2) $J_0^\alpha D_C^\alpha\omega=\omega$;

(3) $\omega^{(i)}(x)\in C[0,1](i=0,1,2,\cdots,n-1)$.

Proof. (1) $\omega\in J_0^\alpha C$ implies that there exists a $\omega_1\in C$ such that $\omega=J_0^\alpha\omega_1$. Noting that for every $i=0,1,\cdots,n-1$, one has

$$|J_0^{\alpha-i}\omega_1(x)|=\left|\frac{1}{\Gamma(\alpha-i)}\int_0^x(x-t)^{\alpha-i-1}\omega_1(t)\mathrm{d}t\right|$$
$$\le\frac{1}{\Gamma(\alpha-i)}\|\omega_1\|_C\int_0^x(x-t)^{\alpha-i-1}\mathrm{d}t$$
$$=\frac{1}{\Gamma(\alpha-i+1)}\|\omega_1\|_C x^{\alpha-i}\to0\quad(x\to0).$$

So $\omega^{(i)}(0)=D_C^iJ_0^\alpha\omega_1(x)|_{x=0}=J_0^{\alpha-i}\omega_1(x)|_{x=0}=0$. Besides, using $\omega_1\in C[0,1]$ and the third assertion of Property 2.1, $D_C^\alpha\omega=D_C^\alpha J_0^\alpha\omega_1=\omega_1\in C[0,1]$.

(2) $J_0^\alpha D_C^\alpha\omega=J_0^\alpha(D_C^\alpha J_0^\alpha\omega_1)=J_0^\alpha\omega_1=\omega$.

(3) Using (2), (1) of this Property and Lemma 2.1, one gets

$\omega^{(i)}(x)=D^i\omega=D_C^i(J_0^\alpha D_C^\alpha\omega)=J_0^{\alpha-i}D_C^\alpha\omega\in C[0,1]$,

$i=0,1,2,\cdots,n-1$.

Now, defining the space of the exact solution: the fractional C^αspace.

Definition 2.3.

Define $C^\alpha[0,1]=J_0^\alpha C$ equipped with the norm

$$\|u\|_\alpha=\max\left\{\left\|u^{(i)}\right\|_C,\left\|D_C^\alpha u\right\|_C,i=0,1,2,\cdots,n-1\right\},\forall u\in C^\alpha[0,1]$$

Lemma 2.3.

$C^\alpha$is a Banach Space.

Proof. It's easy to prove that C^α is a normed linear space. So we only need to prove the completeness of C^α which implies that C^α is a Banach space. In fact, assuming that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in C^α. Then $u_k^{(i)}\left(i=0,1,\cdots,n-1\right)$ and $D_C^\alpha u_k$ are all Cauchy sequences in C. Hence, there exist $u^i\left(i=0,1,\cdots,n-1\right)$ and $v\in C$ such that

$$\| u_k^{(i)}-u^i \|_C \left(i=0,1,\cdots,n-1\right) \qquad \| D_C^\alpha u_k - v \|_C \to 0\left(k\to\infty\right)$$

Setting $v_1=J_0^\alpha v$, then $v_1\in C^\alpha$ and using Lemma 2.1 and Property 2.2, we obtain that for each

$$|u_k^i-v_1^{(i)}|=|D^iJ_0^\alpha D_C^\alpha(u_k-v_1)|=|J_0^{\alpha-i}D_C^\alpha(u_k-v_1)|\le\frac{1}{\Gamma(\alpha-i+1)}\| D_C^\alpha u_k-v\|_C.$$

Thus $\| u_k^{(i)}-v_1^{(i)}\|_C\to0$ as $k\to\infty$. This implies $u^i=v_1^{(i)}$ and

$$\begin{aligned}&\| u_n-v_1\|_\alpha\\ =&\max\{\| u_k^{(i)}-v_1^{(i)}\|_C,\ \| D_C^\alpha u_k-D_C^\alpha v_1\|_C, i=0,1,\cdots,n-1\}\\ =&\max\{\| u_k^{(i)}-u^i\|_C,\ \| D_C^\alpha u_k-v\|_C, i=0,1,\cdots,n-1\}\to0\ \left(k\to\infty\right).\end{aligned}$$

Therefore, C^α is the complete space.

Denoting by X the set of all polynomials, then X is a dense subset of $C\left[0,1\right]$.

Defining $J_0^\alpha X=\left\{J_0^\alpha p\,\middle|\,p\in X\right\}$.

Lemma 2.4.

$J_0^\alpha X$ is a dense subset of C^α.

Proof. Obviously, $J_0^\alpha X\subset J_0^\alpha C=C^\alpha$. We next prove that $J_0^\alpha X$ is dense in C^α. For arbitrary $u\in C^\alpha$, then $D_C^\alpha u\in C[0,1]$. For any $\varepsilon>0$, there exists a $p\in X$ such that

$$\| D_C^\alpha u-p\|_C<\varepsilon.$$

Setting $q(x)=J_0^\alpha p(x)\in J_0^\alpha X$ and using Lemma 2.1, we obtain that

$$\| D_C^\alpha u-D_C^\alpha q\|_C=\| D_C^\alpha u-p\|_C<\varepsilon,$$

$$\| u^{(i)}-q^{(i)}\|_C=\| D^iJ_0^\alpha D_C^\alpha(u-q)\|_C=\| J_0^{\alpha-i}D_C^\alpha(u-q)\|_C$$

$$\le\| D_C^\alpha u-D_C^\alpha q\|_C\frac{1}{\Gamma(\alpha-i+1)}<\frac{\varepsilon}{\Gamma(\alpha-i+1)},$$

where i = 0, 1, · · · , n − 1. Thus

$$\| u-q\|_\alpha=\max\{\| D_C^\alpha u-D_C^\alpha q\|_C,\ \| u^{(i)}-q^{(i)}\|_C, i=0,1,\cdots,n-1\}$$

$$\le\varepsilon\max\{1,\frac{1}{\Gamma(\alpha-i+1)}, i=0,1,\cdots,n-1\}=M\varepsilon.$$

That is,

$$\exists\, p\in X\Rightarrow\| u-J_0^\alpha p\|_\alpha\le M\varepsilon.$$

Hence, $J_0^\alpha X$ is a dense subset of C^α.

3. ε-approximate Solution

3.1 ε-approximate Solution of Equation (3)

In this subsection, we will give a new collocation method based on the new C [0, 1]. We first give a definition.

Definition 3.1.

Assuming that $\omega\in C^\alpha\left[0,1\right]$. If $\left\|\mathbb{L}\omega-f\right\|_C=\max\limits_{0\le x\le1}\left|\mathbb{L}\omega\left(x\right)-f\left(x\right)\right|\le\varepsilon$ for some small positive number ε, then we say that $\omega\left(x\right)$ is an ε-approximate solution of Equation (3).

Lemma 3.1.

$\mathbb{L}$ defined by (3) is a bounded linear operator from C^α to C.

Proof. For arbitrary $u\in C^\alpha$. We prove that $\mathbb{L}u\in C$. Indeed, $u(x)\in C$ implies that $D_C^\alpha u\in C$, firstly. Using Lemma 2.1, one gets

$$\int_0^x(x-s)^{-\beta}u(s)\mathrm{d}s=\Gamma(1-\beta)J_0^{1-\beta}u(x)=\Gamma(1-\beta)J_0^{1+\alpha-\beta}D_C^\alpha u(x)\in C,$$

$$u=J_0^\alpha D_C^\alpha u\in C.$$

Hence, $\mathbb{L}u\in C$. Obviously, $\mathbb{L}:C^\alpha\to C$ is linear. We next prove that $\mathbb{L}$ is bounded. In fact, for any

$$\forall u\in C^\alpha, \| D_C^\alpha u\|_C\le\| u\|_\alpha,$$

$$\| p(x)u\|_C\le\| p\|_C\| J_0^\alpha D_C^\alpha u\|_\alpha\le\| p\|_C\| D_C^\alpha u\|_C\frac{1}{\Gamma(1+\alpha)}\le\| p\|_C\| u\|_\alpha\frac{1}{\Gamma(1+\alpha)}$$

and

$$\left\|\int_0^x(x-s)^{-\beta}u(s)\mathrm{d}s\right\|_C=\|\Gamma(1-\beta)J_0^{1+\alpha-\beta}D_C^\alpha u(x)\|_C$$

$$\le\Gamma(1-\beta)\| D_C^\alpha u(x)\|\frac{1}{\Gamma(2+\alpha-\beta)}\le\frac{\Gamma(1-\beta)}{\Gamma(2+\alpha-\beta)}\| u(x)\|_\alpha.$$

Then

$$\|\mathbb{L}u\|_C\le\| D_C^\alpha u\|_C+\| p(x)u\|_C+\left\|\int_0^x(x-s)^{-\beta}u(s)\mathrm{d}s\right\|_C$$

$$\le\| u\|_\alpha\left(1+\frac{\| p\|_C}{\Gamma(1+\alpha)}+\frac{\Gamma(1-\beta)}{\Gamma(2+\alpha-\beta)}\right).$$

which implies that $\mathbb{L}$ is bounded.

Assuming that $\left\{\omega_k\right\}_{k=0}^\infty$ is linearly independent and $X=span\left\{\omega_k,k=0,1,2,\cdots\right\}$. Setting $h_k=J_0^\alpha\omega_k, k=0,1,2,\cdots$.

Theorem 3.1.

For every $\varepsilon>0$, there exists a positive integer N, such that for every $n\ge N$, $v_n\left(x\right)=\sum_{k=0}^n c_k^*h_k\left(x\right)$ is an $\varepsilon-$ approximate solutionnof Equation (3), where c_k^* satisfies

$$\left\|\sum_{k=0}^n c_k^*f_k\left(x\right)-f\left(x\right)\right\|_C=\min_{c_k}\left\|\sum_{k=0}^n c_kf_k\left(x\right)-f\left(x\right)\right\|_C$$

and $f_k\left(x\right)=\mathbb{L}h_k\left(x\right), k=0,1,2,\cdots,n.$

Proof. Letting $v(x)$ be the true solution of Equation (3). Using Lemma 2.4 and Lemma 3.1, there exists an element $w\triangleq\sum_{k=0}^N c_kh_k(x)\in C^\alpha$ satisfying $\| w-v\|_\alpha\le\frac{\varepsilon}{\|\mathbb{L}\|}$. Thus

$$\|\mathbb{L}w-f\|_C=\|\mathbb{L}w-\mathbb{L}v\|_C\le\|\mathbb{L}\|\cdot\| w-v\|_\alpha\le\|\mathbb{L}\|\frac{\varepsilon}{\|\mathbb{L}\|}=\varepsilon.$$

That is, $\|\sum_{k=0}^N c_kf_k(x)-f(x)\|_C\le\varepsilon$, where $f_k(x)=\mathbb{L}h_k(x)$, $k=0,1,2,\cdots,n.$ As a result, for every $n\ge N$,

$\|\sum_{k=0}^n c_k^*f_k(x)-f(x)\|_C=\min_{c_k}\|\sum_{k=0}^n c_kf_k(x)-f(x)\|_C\le\varepsilon.$ This assures

$$\left\|\mathbb{L}\sum_{k=0}^n c_k^*h_k(x)-f(x)\right\|_C=\left\|\sum_{k=0}^n c_k^*f_k(x)-f(x)\right\|_C\le\varepsilon.$$

Consequently, Definition 3.1 implies that for every $n \geq N, v_n^*(x) = \sum_{k=0}^{n} c_k^* h_k(x)$ is an ε-approximate solution of Equation (3).

3.2 Solution of Equation (4)

In this subsection, we change the problem of searching the minimum value of Equation (4) into solving a system of linear equations which has one solution at least. And every solution of this system gives an ε-approximate solution of Equation (4). To solve Equation (4), we will replace Equation (4) with

$$\min_{c_i} \sum_{j=0}^{m} \left(\sum_{i=0}^{n} c_i f_i\left(x_j\right) - f\left(x_j\right) \right)^2, \tag{5}$$

where $\left\{ x_j = \frac{1}{m} j \right\}_{j=0}^{m}$.

Set

$f_i = \left(f_i\left(x_0\right), f_i\left(x_1\right), \cdots, f_i\left(x_m\right) \right)^{\mathrm{T}} \in \mathbb{R}^{m+1}$,

$F_1 = \left(f\left(x_0\right), f\left(x_1\right), \cdots, f\left(x_m\right) \right)^{\mathrm{T}} \in \mathbb{R}^{m+1} \left(i = 0,1,2,\cdots,n \right)$

$\mathbb{R}^{m+1}$ is the Euclidean space of dimension $m+1$.Then Equation (4) can be replaced with

$$\min_{c_i \in \mathbb{R}} \left\| \sum_{i=0}^{n} c_i f_i - F_1 \right\|_{\mathbb{R}^{m+1}}^2 . \tag{6}$$

Definition 3.2.

Letting $W = span\left\{ f_0, f_1, \cdots, f_n \right\}$. We call $\phi \in W$ the best approximation element of F_1 in W , if

$\left\| \phi - F_1 \right\| = \min_{S^* \in W} \left\| S^* - F_1 \right\|_{\mathbb{R}^{m+1}}$.

Similar to [7], we can prove the following two theorems.

Theorem 3.2.

There is a unique vector $\phi \in W = span\left\{ f_0, f_1, \cdots, f_n \right\}$ such that

$\left\| \phi - F_1 \right\| = \min_{S^* \in W} \left\| S^* - F_1 \right\|_{\mathbb{R}^{m+1}}$.

Theorem 3.2 shows that we can obtain an ε-approximate solution at least.

Theorem 3.3.

Let $F_1 \in \mathbb{R}^{m+1}$, then $y \in W$ is the best approximation element of F_1 in W if and only if

$\left(F_1 - y, f_k \right) = 0, k = 0,1,2,\cdots,n$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^{m+1}$.

Theorem 3.4.

$y = \sum_{i=0}^{n} c_i^* f_i$ is the best approximation element of (6) with respect to F_1 in W if and only if $\left\{ c_i^* \right\}_{i=0}^{n}$ is the solution of the following equations which is the normal equations of (5).

$$\frac{\partial}{\partial c_k} \sum_{j=0}^{m} \left(\sum_{i=0}^{n} c_i f_i\left(x_j\right) - f\left(x_j\right) \right)^2 = 0, k = 0,1,2,\cdots,n \tag{7}$$

Proof. The system of Equation (7) can be rewritten as

$\sum_{i=0}^{n} c_i (f_i, f_k) = (F_1, f_k), k = 0,1,2,\cdots,n$

which is equivalent to

$(F_1 - y, f_k) = 0, k = 0,1,2,\cdots,n$

So the conclusion follows from Theorem 3.3.

Remark 1.

Using Theorem 3.4, we get that each minimal point of $\sum_{j=0}^{m} \left(\sum_{i=0}^{n} c_i f_i\left(x_j\right) - f\left(x_j\right) \right)^2$ is the minimum point of it. In order to find $c_k\left(k = 0,1,2,\cdots,n\right)$,we can solve Equation (7) or search the minimum value of Equation (5) directly. When n is large, Equation (7) may be ill-conditioned. At this point, searching the minimum value of Equation (5) directly is a good choice since it may partly overcome the ill-conditioned problems.

5. Conclusions

In this paper, a collocation method based on fractional polynomials is proposed for solving fractional integro-di erential equations with the weakly singular kernel. We propose the concept of fractional C^α space and obtain a dense subset of it. A strict theory for obtaining the ε-approximate solution is established. Less amount of calculation can produce precision satisfying the application requirements. The final examples demonstrate the e ciency of the proposed method.

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References

[1] B.Q. Tang, X.F. Li, Solution of a class of Volterra integral equations with singular and weakly singular kernels, Appl. Math. Comput. 199 (2008) 406-413.

[2] V.V. Zozulya, P.I. Gonzalez-Chi, Weakly singular, singular and hypersingular integrals in 3-D elas-ticity and fracture mechanics, J. Chin. Inst. Eng. 22 (1999) 763-775.

[3] S. Momani, A. Jameel, S. Al-Azawi, Local and global uniqueness theorems on fractional integro-di erential equations via Biharis and Gronwalls inequalities, Soochow J. Math. 33 (2007) 619-627.

[4] J. Zhao, J. Xiao, N.J. Ford, Collocation methods for fractional integro-di erential equations with weakly singular kernels, Numer. Algorithms 65 (2014) 723-743.

[5] S. Nemati, S. Sedaghat, I. Mohammadi, A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-di erential equations with weakly singular kernels, J. Comput. Appl. Math. 308 (2016) 231-242.

[6] K. Diethelm, The analysis of fractional di erential equations, Lecture Notes in Mathematics, 2004.

[7] Q.J. Yan, Numerical analysis, Beihang University Press, 2001.